THE PROBABILITY DISTRIBUTION OF THE DISTANCE BETWEEN TWO RANDOM POINTS IN A BOX.

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ABSTRACT. We determine the probability distribution for the distance between two random points in a box with sides a, b, and c. The average of this distance is known before as Robbins's constant.

1. INTRODUCTION

We consider the random variable V = the distance between two random points in a box with sides a, b, and c. V is one of the many random quantities studied in Geometric Probability, see eg. [6]. The classical Sylvester's problem considers the area of the convex hull of three random points in a convex set, in particular in a triangle, a square, or a circle. The three-dimensional variants of these problems have also been studied. These problems, which consider a random area in an area or a random volume in a volume are affinely invariant, meaning that the results are the same for a cube and for a box with different sides and even for a parallelipiped. This is not the case in our present problem which considers a one-dimensional length in a three-dimensional set, explaining why our result depends on the box sides a, b, and c.

The similar problem of the distance between two random points on the surface of a cube has been considered by Borwain et.al. [1] and Philip, [4]. See also Bailey et.al. [2].

The expected value of the distance between two random points in a box was stated as a problem by D.P. Robbins, [5], and calculated by T.S. Bolin, [3]. The distribution function $K(v) = \mathrm{Prob}(V \le v)$ for this random variable seems not to be known before.

A survey of the results in this area are collected by E. Weisstein on the web site [7].

Our method is to determine the distance distributions for each ot the three directions and convolve them to get the distribution of V. The result is a long expression and we rely heavily on the use of a formula manipulating program, in our case *Maple 10*.
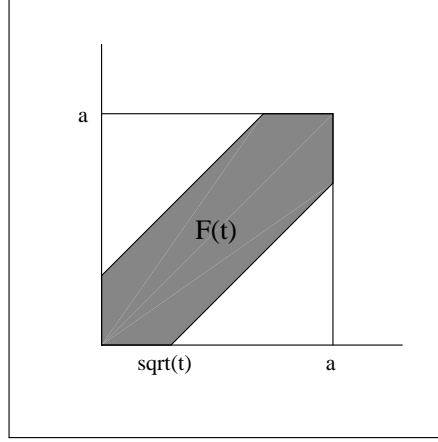
FIGURE 1. The area $F_a(t)$.

2. NOTATION AND FORMULATION.

Let the two random points be (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2). We assume that X_1 and X_2 are independent and evenly distributed in the interval $(0, a)$. The same is assumed for Y_1 and Y_2 in $(0, b)$ and for Z_1 an Z_2 in $(0, c)$. We assume also, without loss of generality, that the box sides satisfy $0 < a \leq b \leq c$.

We start by calculating the distribution function $F_a(t) = \text{Prob}((X_1 - X_2)^2 \leq t)$ and from it the corresponding density function $f_a(t) = \frac{dF_a(t)}{dt}$. Then, the density $g(s)$ corresponding to $G(s) = \text{Prob}((X_1 - X_2)^2 + (Y_1 - Y_2)^2 \leq s)$ is obtained by convolving f_a and f_b. At last, the density $h(u)$ corresponding to $H(u) = \text{Prob}((X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \leq u)$ is the convolution of f_c and g. The wanted distribution function for the distance is $K(v) = H(v^2)$ with the density $k(v) = 2 v h(v^2)$.

3. THE ONE-DIMENSIONAL DISTRIBUTION.

The probability that $(X_1 - X_2)^2 \leq t$ is proportional to the area of the diagonal strip in Fig.1.

$$(1) \qquad F_a(t) = \begin{cases} 1 - (1 - \sqrt{t}/a)^2, & 0 < t \leq a^2 \ ; \\ 1 & a^2 < t. \end{cases}$$

The density is

$$(2) \qquad f_a(t) = \frac{1}{a\sqrt{t}} - \frac{1}{a^2}, \quad 0 < t \leq a^2 \ .$$

When we give the value of a function like $f_a(t)$ in an interval, we tacitly assume that it is zero where it is not defined.

4. THE TWO-DIMENSIONAL DISTRIBUTION.

The probability density for the event $(X_1 - X_2)^2 + (Y_1 - Y_2)^2 \leq s$ is the convolution g of f_a and f_b

$$g(s) = \int f_a(s - t)\, f_b(t) dt.$$

Because of the different domains of definition of f_a and f_b, there are three cases

$$(3) \qquad g_1(s) = \int_0^s f_a(s - t)\, f_b(t) dt, \quad 0 < s \leq a^2 .$$

$$(4) \qquad g_2(s) = \int_{s-a^2}^s f_a(s - t)\, f_b(t) dt, \quad a^2 < s \leq b^2 .$$

$$(5) \qquad g_3(s) = \int_{s-a^2}^{b^2} f_a(s - t)\, f_b(t) dt, \quad b^2 < s \leq a^2 + b^2 .$$

.

If $a = b$, g_2 is omitted. We get

$$(6) \quad g(s) = \begin{cases} -2\frac{\sqrt{s}}{a^2 b} - 2\frac{\sqrt{s}}{ab^2} + \frac{\pi}{ab} + \frac{s}{a^2 b^2}, & 0 < s \leq a^2 ; \\ -2\frac{\sqrt{s}}{a^2 b} \\ \quad -\frac{1}{b^2} + \frac{2}{ab}\arcsin\left(\frac{a}{\sqrt{s}}\right) + \frac{2}{a^2 b}\sqrt{s - a^2}, & a^2 < s \leq b^2 ; \\ -\frac{1}{b^2} + \frac{2}{ab}\arcsin\left(\frac{a}{\sqrt{s}}\right) + \frac{2}{a^2 b}\sqrt{s - a^2} \\ \quad -\frac{1}{a^2} + \frac{2}{ab}\arcsin\left(\frac{b}{\sqrt{s}}\right) + \frac{2}{ab^2}\sqrt{s - b^2} \\ \quad -\frac{\pi}{ab} - \frac{s}{a^2 b^2}, & b^2 < s \leq a^2 + b^2. \end{cases}$$

Since s is the the square of the distance, we get the density for the distance $v = \sqrt{s}$ between two random points in a rectangle with sides a and b as $g_v(v) = g(v^2) \frac{ds}{dv} = 2\, v\, g(v^2)$. This density is shown in Figure 2.

The expectation of the distance between two random points in a rectangle is

$$(7) \qquad E_{\text{rectangle}} = \int_0^{a^2 + b^2} \sqrt{s}\, g(s)\, ds = \int_0^{\sqrt{a^2 + b^2}} v\, g_v(v)\, dv.$$
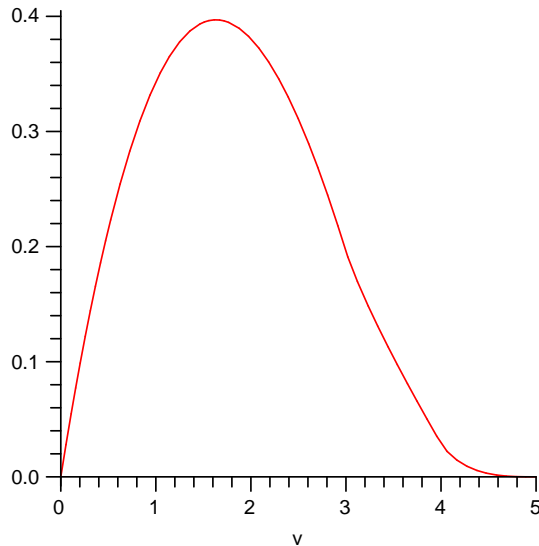
The result is

FIGURE 2. The density function $g_v(v)$ for the distance between two random points in a rectangle with sides 3 and 4.

$$(8) \quad E_{\text{rectangle}} = \frac{a^2}{6\,b} \ln\left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right) + \frac{b^2}{6\,a} \ln\left(\frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}}\right)$$
$$+ \frac{1}{15}\left(\frac{b^3}{a^2} + \frac{a^3}{b^2} + \left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\sqrt{a^2 + b^2}\right)$$

For $a = b = 1$, this reduces to

$$(9) \qquad E_{\text{unit square}} = \frac{1}{3}\ln\left(1 + \sqrt{2}\right) + \frac{1}{15}\left(2 + \sqrt{2}\right) \approx .52140543.$$

5. THE THREE-DIMENSIONAL DISTRIBUTION.

The density of the probability that $(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \leq u$ is the convolution of f_c and g

$$(10) \qquad\qquad h(u) = \int f_c(u - s)\, g(s)\, ds.$$

We shall convolve f_c with each of the three components of g given in (6). Since the components have different domains of definition, this will also be the case for the convolutions. A closer study of (6) motivates the defining of the following g_{ij}.

(11)

$$g_{11} = -2\,\frac{\sqrt{s}}{a^2 b}, \qquad\qquad\qquad\qquad\qquad 0 < s \leq b^2;$$

$$g_{12} = -2\,\frac{\sqrt{s}}{a\,b^2} + \frac{\pi}{a\,b} + \frac{s}{a^2 b^2}, \qquad\qquad\qquad 0 < s \leq a^2;$$

$$g_{22} = -\frac{1}{b^2} + \frac{2}{a\,b}\,\arcsin\left(\frac{a}{\sqrt{s}}\right) + \frac{2}{a^2\,b}\sqrt{s-a^2}, \quad a^2 < s \leq a^2 + b^2;$$

$$g_{32} = -\frac{1}{a^2} + \frac{2}{a\,b}\,\arcsin\left(\frac{b}{\sqrt{s}}\right) + \frac{2}{a\,b^2}\sqrt{s-b^2}, \quad b^2 < s \leq a^2 + b^2;$$

$$g_{33} = -\frac{\pi}{a\,b} - \frac{s}{a^2 b^2}, \qquad\qquad\qquad\qquad b^2 < s \leq a^2 + b^2.$$

Then,

$$(12) \qquad \begin{aligned} g_1 &= g_{11} + g_{12}, & 0 < s \leq a^2; \\ g_2 &= g_{11} + g_{22}, & a^2 < s \leq b^2; \\ g_3 &= g_{22} + g_{32} + g_{33}, & b^2 < s \leq a^2 + b^2. \end{aligned}$$

With this splitting, we can write g as the sum

$$(13) \qquad\qquad g = g_{11} + g_{12} + g_{22} + g_{32} + g_{33},$$

Convolving each of the g_{ij} with f_c ,

$$(14) \qquad\qquad h_{ij}(u) = \int f_c(u-s)\,g_{ij}(s)\,ds.$$

we can write

$$(15) \qquad\qquad h = h_{11} + h_{12} + h_{22} + h_{32} + h_{33},$$

Notice that g_{32} is g_{22} with a and b switched. This applies also to their boundaries of definition implying that we can get h_{32} by switching a and b in h_{22}.

A detailed description of the resulting $h(u)$ is given in the next section.

Like g, each h_{ij} has different analytical expressions h_{ijk} in three adjacent intervals. Remembering that $a \leq b \leq c$, and looking at e.g. h_{11}, we have

$$(16) \qquad\qquad h_{11} = \begin{cases} h_{111}, & 0 < u \leq b^2; \\ h_{112}, & b^2 < u \leq c^2; \\ h_{113}, & c^2 < u \leq b^2 + c^2. \end{cases}$$
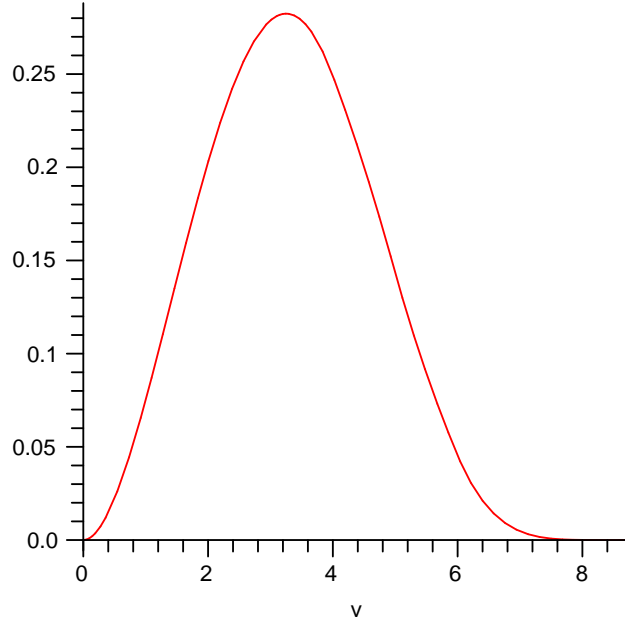
FIGURE 3. Density function $k(v)$ for the distance be-
tween two random points in a box with sides 4, 5, and
6.

Altogether, there are eight breakpoints on the u-axis for the analyt-
ical description of $h(u)$ and they are

$$(0, a^2, b^2, c^2, a^2 + b^2, a^2 + c^2, b^2 + c^2, a^2 + b^2 + c^2)$$

.

If $c^2 < a^2 + b^2$ the points above are in increasing order. Otherwise,
points 4 and 5 above should change place to show the intervals for the
analytical expressions. This implies that we have two cases if we want
to give $h(u)$ for each interval. Writing $h(u)$ as in (15) circumvents this
difficulty. If $a = b$ or $b = c$ the number of intervals decreases to five
and if they are all alike there are just three intervals left.

6. THE DENSITY OF THE SQUARE OF THE DISTANCE IN A BOX.

The practical integrations of type (14) are made with *Maple 10*. The
Maple worksheet habc.mw is available at www.math.kth.se/~johanph.
We give the result here as functions of u , which is the square of
distance between two random points in a box with sides a, b, and c.
The density $k(v)$ for the distance $v = \sqrt{u}$ is $k(v) = 2\,v\,h(v^2)$ and is
shown in Figure 3.

(17)
$$h_{11} = \frac{1}{3\,a^2\,b^2\,c^2}.$$

$$\begin{cases} -3\,\pi b\,c\,u + 4\,b\,u^{3/2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < u \le b^2, \\[2mm] 4\,b^4 + 6\,b^2 c\sqrt{u-b^2} - 6\,b\,c\,u\arcsin(\tfrac{b}{\sqrt{u}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad b^2 < u \le c^2, \\[2mm] 4\,b^4 + 6\,b^2\,c\,\sqrt{u-b^2} \\ +6\,b\,c\,u(\arccos(\tfrac{c}{\sqrt{u}}) - \arcsin(\tfrac{b}{\sqrt{u}})) \\ -2\,b\,(2\,u + c^2)\sqrt{u-c^2}. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad c^2 < u \le b^2 + c^2. \end{cases}$$

(18)
$$h_{12} = \frac{1}{6\,a^2\,b^2\,c^2}.$$

$$\begin{cases} 12\,\pi a\,b\,c\sqrt{u} - 6\,\pi\,a(b+c)u + 8(a+c)\,u^{3/2} - 3\,u^2, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < u \le a^2, \\[2mm] 5\,a^4 - 6\,\pi\,a^3 b, \\ +12\,\pi a\,b\,c\sqrt{u} + 8\,c\,u^{3/2} \\ -12\,\pi\,a\,b\,c\,\sqrt{u-a^2} - 8\,c\,(u-a^2)^{3/2} \\ -12\,a\,c\,u\arcsin(\tfrac{a}{\sqrt{u}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad a^2 < u \le c^2, \\[2mm] 5\,a^4 - 6\,\pi\,a^3 b + 6\,\pi\,abc^2 - c^4 + 6(\,\pi\,a\,b + c^2)\,u + 3\,u^2 \\ -12\,\pi\,a\,b\,c\,\sqrt{u-a^2} - 8\,c\,(u-a^2)^{3/2} \\ -4\,a(2\,u + c^2)\sqrt{u-c^2} \\ +12\,a\,c\,u(\arccos(\tfrac{c}{\sqrt{u}}) - \arcsin(\tfrac{a}{\sqrt{u}})), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad c^2 < u \le a^2 + c^2. \end{cases}$$

(19)

$$h_{22} = \frac{1}{3\,a^2\,b^2\,c^2}.$$

$$
\begin{cases}
0\,, \\
\qquad\qquad\qquad\qquad\qquad\qquad 0 < u \le a^2, \\[2mm]
3\,\pi\,a^2\,b(a+c) - 3\,a^4 - 6\,\pi\,a\,b\,c\,\sqrt{u} + 3\,(a^2+\pi b c)u \\
+(6\,\pi abc - 2(b+3c)a^2 - 4\,bu)\sqrt{u-a^2} \\
-6\,a\,b\,u\,\arcsin(\tfrac{a}{\sqrt{u}}), \\
\qquad\qquad\qquad\qquad\qquad a^2 < u \le a^2 + b^2, \\[2mm]
3\,a^2 b(\pi\,a - b) - 4\,b^4 \\
-12\,a\,b\,c\,\arcsin(\tfrac{b\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-a^2}})\,\sqrt{u} \\
-6\,a\,c\,(a-\pi b)\,\sqrt{u-a^2} \\
-6\,c\left(b^2 - a^2 + 2ab\arcsin(\tfrac{a}{\sqrt{a^2+b^2}})\right)\sqrt{u-a^2-b^2} \\
-6\,a\,b\,(a^2+b^2)\arcsin(\tfrac{a}{\sqrt{a^2+b^2}}) \\
+6\,b\,c\,(a^2+u)\arcsin\left(\tfrac{b}{\sqrt{u-a^2}}\right), \\
\qquad\qquad\qquad\qquad a^2 + b^2 < u \le a^2 + c^2, \\[2mm]
3\,a^2(a^2 - b^2 - c^2) - 4\,b^4 - 3\,a^2\,u \\
-12\,a\,b\,c\left(\arcsin(\tfrac{b\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-a^2}}) - \arccos(\tfrac{ac}{\sqrt{u-c^2}\sqrt{u-a^2}})\right)\sqrt{u} \\
+2\,b\,(a^2+c^2+2\,u)\sqrt{u-a^2-c^2} \\
-6\,c(b^2 - a^2 + 2\,ab\,\arcsin(\tfrac{a}{\sqrt{a^2+b^2}}))\sqrt{u-a^2-b^2} \\
-6\,a\,b\,(a^2+b^2)\arcsin(\tfrac{a}{\sqrt{a^2+b^2}}) \\
+6\,b\,c\,(a^2+u)\left(\arcsin(\tfrac{b}{\sqrt{u-a^2}}) - \arccos(\tfrac{c}{\sqrt{u-a^2}})\right) \\
+6\,ab\,(c^2+u)\arcsin(\tfrac{a}{\sqrt{u-c^2}}), \\
\qquad\qquad\qquad\qquad a^2 + c^2 < u \le a^2 + b^2 + c^2
\end{cases}
$$

(20)

$$h_{32} = \frac{1}{3\,b^2\,a^2\,c^2} \cdot$$

$$
\begin{cases}
0, & 0 < u \le b^2, \\[6pt]
\begin{aligned}
&3\,\pi\,a\,b^2\,(b+c) - 3\,b^4 - 6\,\pi\,a\,b\,c\,\sqrt{u} + 3\,(b^2 + \pi a\,c)u \\
&+(6\,\pi\,a\,b\,c - 2(a+3\,c)b^2 - 4\,a\,u)\sqrt{u - b^2} \\
&-6\,a\,b\,u\,\arcsin(\tfrac{b}{\sqrt{u}}),
\end{aligned} & b^2 < u \le a^2 + b^2, \\[6pt]
\begin{aligned}
&3\,a\,b^2(\pi\,b - a) - 4\,a^4 \\
&-12\,a\,b\,c\arcsin(\tfrac{a\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-b^2}})\sqrt{u} \\
&-6\,bc\,(b - \pi\,a)\,\sqrt{u - b^2} \\
&-6\,c\left(a^2 - b^2 + 2\,a\,b\arcsin(\tfrac{b}{\sqrt{a^2+b^2}})\right)\sqrt{u - a^2 - b^2} \\
&-6\,a\,b\,(a^2 + b^2)\arcsin(\tfrac{b}{\sqrt{a^2+b^2}}) \\
&+6\,a\,c\,(b^2 + u)\arcsin\left(\tfrac{a}{\sqrt{u-b^2}}\right),
\end{aligned} & a^2 + b^2 < u \le b^2 + c^2, \\[6pt]
\begin{aligned}
&3\,b^2\,(b^2 - a^2 - c^2) - 4\,a^4 - 3\,b^2\,u \\
&-12\,a\,b\,c\left(\arcsin(\tfrac{a\sqrt{u}}{\sqrt{a^2+b^2}\sqrt{u-b^2}}) - \arccos(\tfrac{bc}{\sqrt{u-c^2}\sqrt{u-b^2}})\right)\sqrt{u} \\
&+2\,a\,(b^2 + c^2 + 2\,u)\,\sqrt{u - b^2 - c^2} \\
&-6\,c\,(a^2 - b^2 + 2\,a\,b\,\arcsin(\tfrac{b}{\sqrt{a^2+b^2}}))\sqrt{u - a^2 - b^2} \\
&-6\,a\,b\,(a^2 + b^2)\arcsin(\tfrac{b}{\sqrt{a^2+b^2}}) \\
&+6\,a\,c\,(b^2 + u)\left(\arcsin(\tfrac{a}{\sqrt{u-b^2}}) - \arccos(\tfrac{c}{\sqrt{u-b^2}})\right) \\
&+6\,a\,b\,(c^2 + u)\arcsin(\tfrac{b}{\sqrt{u-c^2}}),
\end{aligned} & b^2 + c^2 < u \le a^2 + b^2 + c^2
\end{cases}
$$

(21)

$$h_{33} = \frac{1}{6\,a^2\,b^2\,c^2}.$$

$$\begin{cases} 0\,, & 0 < u \leq b^2\,, \\[2mm] 3\,(2\,\pi\,a\,b + b^2 + u)(u - b^2) \\ -4c\,(b^2 + 3\,\pi\,ab + 2\,u)\sqrt{u - b^2}, \\[2mm] & b^2 < u \leq a^2 + b^2, \\[2mm] 3\,(a^2 + b^2)^2 - 3\,b^4 + 6\,\pi\,a^3 b \\ -4\,c\,(b^2 + 3\pi\,a\,b + 2\,u)\sqrt{u - b^2} \\ +4\,c\,(a^2 + b^2 + 3\,\pi\,ab + 2\,u)\sqrt{u - a^2 - b^2}, \\[2mm] & a^2 + b^2 < u \leq b^2 + c^2, \\[2mm] 3\,(a^2 + b^2)^2 + c^4 + 6\,\pi\,a\,b\,(a^2 + b^2 - c^2) \\ -6\,(\pi\,a\,b + c^2)\,u - 3\,u^2 \\ +4\,c\,(a^2 + b^2 + 3\,\pi\,ab + 2\,u)\sqrt{u - a^2 - b^2}, \\[2mm] & b^2 + c^2 < u \leq a^2 + b^2 + c^2 \end{cases}$$

Putting $a = b = c = 1$ in h, we get the density for the unit cube which is a reasonably long expression. We give the density $k_{\text{cube}}(v) = 2\,v\,h(v^2)$ for the distance V between two random points in a unit cube.

(22)

$$k_{\text{cube}}(v) = \begin{cases} v^2\,(4\,\pi - 6\,\pi\,v + 8\,v^2 - v^3) \\[2mm] & 0 < v \leq 1, \\[2mm] (6\,\pi - 1)\,v - 8\,\pi\,v^2 + 6\,v^3 + 2\,v^5 + 24\,v^3\,\arctan\sqrt{v^2 - 1} \\ -8\,v\,(1 + 2\,v^2)\sqrt{v^2 - 1} \\[2mm] & 1 < v \leq \sqrt{2}, \\[2mm] (6\,\pi - 5)\,v - 8\,\pi\,v^2 + 6\,(\pi - 1)\,v^3 - v^5 \\ +8\,v\,(1 + v^2)\sqrt{v^2 - 2} \\ -24\,v\,(1 + v^2)\arctan\sqrt{v^2 - 2} + 24\,v^2\,\arctan(v\,\sqrt{v^2 - 2}) \\[2mm] & \sqrt{2} < v \leq \sqrt{3}. \end{cases}$$

7. THE AVERAGE DISTANCE BETWEEN TWO RANDOM POINTS IN A BOX.

This average distance $E(V)$ was given by T.S. Bolis [3] and we present his result in the next section. Here, we shall describe how we have calculated the same average. Of course, the results coincide. Since $v = \sqrt{u}$, we have

$$(23) \qquad E(V) = \int \sqrt{u}\, h(u)\, du = \int \sqrt{u} \int f_c(u - s)\, g(s)\, ds\, du.$$

We start by calculating

$$(24) \qquad m(s) = \int \sqrt{u}\, f_c(u - s)\, du,$$

and get

$$(25) \quad m(s) = \frac{s}{c} \log\left(\frac{c}{\sqrt{s}} + \sqrt{1 + \frac{c^2}{s}} \right) + \frac{2}{3} \frac{s^{3/2}}{c^2} + \frac{1}{3\,c^2}(c^2 - 2\,s)\sqrt{s + c^2}.$$

This function is well defined and increasing from $\frac{c}{3}$ on the interval $0 \leq s < \infty$.

Then, we get $E(V)$ as the sum of five intgrals of the form

$$(26) \qquad E_{ij} = \int m(s)\, g_{ij}(s)\, ds.$$

Moreover, E_{32} is E_{22} with a and b switched.

The second moment of the distance is easily calculated as

$$(27)$$
$$\alpha_2 = \int_0^{a^2} t\, f_a(t)\, dt + \int_0^{b^2} t\, f_b(t)\, dt + \int_0^{c^2} t\, f_c(t)\, dt = \frac{1}{6}\left(a^2 + b^2 + c^2 \right).$$

Our calculations are described in the *Maple* worksheet `Eabc.mw`, which is available at `www.math.kth.se/~johanph`.

8. ROBBINS'S CONSTANT

Robbins stated this problem in [5]. Bolis, [3] presented a solution obtained by splitting the box into three similar cones and integrating over a cone in spherical coordinates.

Notice that the box in [3] has the sides $2a$, $2b$, and $2c$. The result given below is that in [3] divided by two.

Define:

$$(28) \qquad \begin{aligned} r &= \sqrt{a^2 + b^2 + c^2} \\ r_1 &= \sqrt{b^2 + c^2} \\ r_2 &= \sqrt{c^2 + a^2} \\ r_3 &= \sqrt{a^2 + b^2}. \end{aligned}$$

Then, the average can be written in the following symmetric and condensed form

(29)
$$E(V) = \frac{1}{15} r - \frac{7}{90\,a^2} (r - r_1) {r_1}^2 - \frac{7}{90\,b^2} (r - r_2){r_2}^2 - \frac{7}{90\,c^2} (r - r_3){r_3}^2$$
$$+ \frac{4}{315\,a^2\,b^2\,c^2} \left(a^7 + b^7 + c^7 - {r_1}^7 - {r_2}^7 - {r_3}^7 + r^7 \right)$$
$$+ \frac{1}{30\,a\,b^2\,c^2} \left(b^6 \mathrm{arsinh}\left(\frac{a}{b} \right) + c^6 \mathrm{arsinh}\left(\frac{a}{c} \right) - {r_1}^2 \left({r_1}^4 - 8\,b^2 c^2 \right) \mathrm{arsinh}\left(\frac{a}{r_1} \right) \right)$$
$$+ \frac{1}{30\,a^2\,b\,c^2} \left(c^6 \mathrm{arsinh}\left(\frac{b}{c} \right) + a^6 \mathrm{arsinh}\left(\frac{b}{a} \right) - {r_2}^2 \left({r_2}^4 - 8\,c^2 a^2 \right) \mathrm{arsinh}\left(\frac{b}{r_2} \right) \right)$$
$$+ \frac{1}{30\,a^2\,b^2\,c} \left(a^6 \mathrm{arsinh}\left(\frac{c}{a} \right) + b^6 \mathrm{arsinh}\left(\frac{c}{b} \right) - {r_3}^2 \left({r_3}^4 - 8\,a^2 b^2 \right) \mathrm{arsinh}\left(\frac{c}{r_3} \right) \right)$$
$$- \frac{2}{15\,a\,b\,c} \left(a^4 \arcsin\left(\frac{bc}{r_2\,r_3} \right) + b^4 \arcsin\left(\frac{ca}{r_3\,r_1} \right) + c^4 \arcsin\left(\frac{ab}{r_1\,r_2} \right) \right).$$

The following relation holds for the arsinh function

(30)
$$\mathrm{arsinh}(y) = \frac{1}{2} \log\left(y + \sqrt{1 + y^2} \right).$$

Inserting $a = b = c = 1$, we get the value for the unit cube

(31)
$$E_{\text{unit cube}} = \frac{1}{105} \left(4 + 17\sqrt{2} - 6\,\sqrt{3} + 21\,\log\left(1 + \sqrt{2} \right) + 84\,\log\left(1 + \sqrt{3} \right) \right.$$
$$\left. -42\,\log\left(2 \right) - 7\,\pi \right) \approx .661707182.$$

9. Comment.

Even if *Maple* is very helpful in doing the calculations of this paper, there are several things it doesn't do. The success of the calculations relies on manual simplification of trigonometric expressions and on manual factorization of polynomials.

We tested our method on the corresponding four-dimensional problems. When trying to calculate the four-dimensional distribution or average distance, we encountered integrals that we cannot solve like

$$\int \frac{\arctan x}{\sqrt{1 - a\,x^2}\,\sqrt{1 - b\,x^2}}\, dx, \text{ where } a > 0 \text{ and } b > 0 .$$

References

[1] D.H. Bailey, J.M. Borwein, V Kapoor, and E.W. Weisstein *Ten Problems in Experimental Mathematics,*
http://crd.lbl.gov/~dhbailey/dhbpapers/tenproblems.pdf, March 8, 2006.

[2] D.H. Bailey, J.M. Borwein, R.E. Crandall *Box integrals,* Journal of Computational and Applied Mathematics Vol. 206, Issue 1, Sept. 2007, pp. 196-208.

[3] T.S. Bolis *Solution of problem E2629: Average Distance between Two Points in a Box,* Amer. Math. Monthly Vol. 85, No. 4. pp. 277-78, April. 1978.

[4] J. Philip, *Calculation of Expected Distance on a Unit Cube.* `www.math.kth.se/~johanph`, Jan 2007.

[5] D.P. Robbins *Problem E2629: Find the Average Distance between Two Points in a Box,* Amer. Math. Monthly [1977,57].

[6] L.A. Santalo, *Integral Geometry and Geometric Probability* Encyclopedia of Mathematics and Its Applications, Addison-Wesley, 1976

[7] E. Weisstein at `www.mathworld.wolfram.com`. Look at Geometry > Computational Geometry > Random Point Picking > .. or at Geometry > Solid Geometry > Polyhedra > .. .

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